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Total Positivity and Convexity Preservation

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In this paper we define convexity and rational convexity preservation of systems of functions and we show that total positivity and rational convexity preservation are equivalent. We also characterize certain convexity preserving systems in terms of weak Tchebycheff systems. Curve intersections and curvatures of Bézier curves are also studied. © 1999 Academic Press

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1. INTRODUCTION

It is well established that systems of totally positive blending functions, such as the Bernstein and B-spline bases, preserve monotonicity and convexity and are generally "shape preserving" [9]. In this paper we show that total positivity is *equivalent* to the preservation of all orders of convexity.

By a *system* we understand a sequence of functions $(u_0, ..., u_n)$ defined on an interval $[a, b]$. Given points $P_0, ..., P_n \in \mathbb{R}^d$, called *control points*, the system generates a curve

$$p(t) = \sum_{i=0}^{n} P_i u_i(t), \qquad t \in [a, b], \tag{1.1}$$

whose *control polygon* is the polygonal arc whose vertices are $P_0, ..., P_n$.

The system $(u_0, ..., u_n)$ is said to be *blending* if $u_i(t) \geqslant 0$, $i = 0, ..., n$, and $\sum_{i=0}^n u_i(t) = 1$. Given *weights* $w_0, ..., w_n > 0$ in addition to points $P_0, ..., P_n \in \mathbb{R}^d$, a blending system also generates a rational curve

$$p(t) = \frac{\sum_{i=0}^n w_i P_i u_i(t)}{w(t)}, \qquad w(t) = \sum_{i=0}^{n} w_i u_i(t), \qquad t \in [a, b]. \tag{1.2}$$

Blending systems have the advantage that the non-rational and rational curves they generate lie in the convex hulls of their control polygons.

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A given system $(u_0, ..., u_n)$ of nonnegative functions whose sum is (strictly) positive can be normalised to a blending system $(\hat{u}_0, ...\hat{u}_n)$ by setting $\hat{u}_i = u_i / \sum_j u_j$.

Generally speaking, a system is said to be "shape preserving" when the curves it generates tend to mimic the shapes of their polygons. This concept is of fundamental importance for the design of curves in geometric modelling. It is relatively recently that some of the classes of systems which possess particular shape preserving properties have been precisely determined [2–6]. Generally these classes of systems include the totally positive ones. Recall that a system $(u_0, ..., u_n)$ defined on $[a, b]$ is *totally positive* if all minors of its collocation matrices

$$M\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_m \end{pmatrix} = (u_j(t_i))_{0 \leqslant i \leqslant m,\, 0 \leqslant j \leqslant n} \tag{1.3}$$

$(a \leqslant t_0 < \cdots < t_m \leqslant b)$ are nonnegative. If all such minors of order up to k are nonnegative then we say that $(u_0, ..., u_n)$ is $\mathrm{TP_k}$. In contrast $(u_0, ..., u_n)$ is said to be *weak Tchebycheff* if all such minors of *precisely* order $n+1$ are nonnegative. We will also say that $(u_0, ..., u_n)$ is $\mathrm{WT_k}$ if all such minors of precisely order k are nonnegative.

Carnicer and Peña [6] characterized monotonicity preserving systems on an interval. It was later shown in [2] that the class of such systems is equivalent to the class of hodograph diminishing systems. Such systems include TP_2 systems. Various forms of convexity preservation have been studied and characterized by Carnicer, García-Esnaola and Peña [3–5]. In particular, in Section 3 of [4], so called "geometrically convexity preserving systems" were defined and analyzed. Furthermore higher order convexity preservation was defined recursively in [4] and its relationship to total positivity studied.

A common assumption on all planar curves considered in [2–5] is that they can be represented as the graphs of univariate functions. In the current paper we study *global* convexity, that is we allow a convex curve to turn through an angle of up to 2π, rather than merely π, and this greater generality simplifies some concepts.

Our basic approach is to take the view that several features of a parametric curve $p(t)$, such as convexity, curvature, torsion, and normal vectors, can be formulated in terms of multilinear alternating functions of points or derivatives of $p(t)$. We therefore make some basic observations concerning multilinear alternating functions in Section 2.

In Section 3 we define d-convexity of curves and we characterize totally positive blending systems in terms of d-convexity. We also derive a necessary condition for convexity preservation which we show to be sufficient when the system has either three or four functions. In Section 4 we both weaken and generalize some conditions in [2] for bounding the number of intersections between two curves generated by totally positive blending systems. In Section 5 we derive some auxiliary results on Wronskians and use them in Section 6 to obtain formulas for alternating functions of sequences of derivatives of Bézier and rational Bézier curves in terms of points in the de Casteljau algorithm. These formulas generalize some found in [8].

2. ALTERNATING FUNCTIONS

A function $\phi: \mathbb{R}^d \times \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$, where $\mathbb{R}^d$ occurs k times, is *multilinear* if it is linear in each variable while the others remain fixed and is *alternating* if $\phi(v_1, ..., v_k) = 0$ whenever $v_i = v_j$, for some $i \neq j$. The set of all such ϕ forms a vector space denoted by $\Omega^k(\mathbb{R}^d)$ (see Spivak [11], p. 280).

For a matrix A of order $m \times n$ we denote by

$$A\begin{pmatrix} i_1, ..., i_p \\ j_1, ..., j_q \end{pmatrix}$$

the submatrix of A consisting of rows $i_1, ..., i_p$ and columns $j_1, ..., j_q$. Given vectors $v_1, ..., v_k \in \mathbb{R}^d$ we let V be the $d \times k$ matrix whose jth column is v_j treated as a column vector $(v_j^1, ..., v_j^d)^T$. Following Ando [1] let $Q_{k,d}$ denote the set of all k-tuples $\alpha = (\alpha_1, ..., \alpha_k)$ such that $1 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k \leqslant d$. It is shown in [11], pp. 280–281, that the dimension of the space $\Omega^k(\mathbb{R}^d)$ is $\binom{d}{k}$ and a basis for the space is given by the functions ϕ_α, $\alpha \in Q_{k,d}$, defined as

$$\phi_\alpha(v_1, ..., v_k) = \det V\begin{pmatrix} \alpha_1, ..., \alpha_k \\ 1, ..., k \end{pmatrix}.$$

We will be particularly concerned with the two specials cases $k = d$ and $k = d-1$. Up to a scalar multiple, the only function in $\Omega^d(\mathbb{R}^d)$ is the determinant function

$$\det(v_1, ..., v_d) := \begin{vmatrix} v_1^1 & \cdots & v_d^1 \\ \vdots & & \vdots \\ v_1^d & \cdots & v_d^d \end{vmatrix}.$$

The dimension of $\Omega^{d-1}(\mathbb{R}^d)$ on the other hand is d. Given vectors $v_1, ..., v_{d-1} \in \mathbb{R}^d$, let us define their *normal vector* n to be

$$n(v_1, ..., v_{d-1}) = \begin{vmatrix} e_1 & v_1^1 & \cdots & v_{d-1}^1 \\ \vdots & \vdots & & \vdots \\ e_d & v_1^d & \cdots & v_{d-1}^d \end{vmatrix},$$

where $e_1, ..., e_d$ is the standard orthonormal basis for $\mathbb{R}^d$, $e_i = (0, ..., 1, 0, ..., 0)^T$, the 1 occurring in the ith position. Then the coordinate functions $n^1, ..., n^d$ of n belong to $\Omega^{d-1}(\mathbb{R}^d)$, for

$$n^i(v_1, ..., v_{d-1}) = (-1)^{i-1} \det V \begin{pmatrix} 1, ..., \hat{\imath}, ..., d \\ 1, ..., d-1 \end{pmatrix},$$

where $\hat{\imath}$ denotes the deletion of i from the sequence 1, ..., d. When $d = 3$, we have the familiar example $n(v_1, v_2) = v_1 \times v_2$, the cross product and, combining it with the scalar product we have the triple product

$$\det(v_1, v_2, v_3) = (v_1 \times v_2) \cdot v_3.$$

We remark also that the *exterior product* of a set of vectors can be identified with multilinear alternating functions [10].

We will regard a given $\phi \in \Omega^k(\mathbb{R}^d)$ as a function operating on *vectors* $v_1, ..., v_k$ and associate with ϕ a related function $\phi' : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \to \mathbb{R}$, with $\mathbb{R}^d$ taken $k+1$ times, acting on *points* $P_0, P_1, ..., P_k$ in $\mathbb{R}^d$, defined as

$$\phi'(P_0, P_1, ..., P_k) = \phi(P_1 - P_0, P_2 - P_1, ..., P_k - P_{k-1}).$$

For example, the signed volume of a d-simplex with vertices $P_0, ..., P_d$ in $\mathbb{R}^d$ is given by

$$\text{vol}(P_0, ..., P_d) = \frac{1}{d!} \det{}'(P_0, ..., P_d). \tag{2.1}$$

Using properties of determinants one can show that for $1 \leqslant k \leqslant d$ and $\alpha \in Q_{k,d}$,

$$(\phi_\alpha)'(P_0, ..., P_k) = \begin{vmatrix} 1 & \cdots & 1 \\ P_0^{\alpha_1} & \cdots & P_k^{\alpha_1} \\ \vdots & & \vdots \\ P_0^{\alpha_k} & \cdots & P_k^{\alpha_k} \end{vmatrix}. \tag{2.2}$$

From (2.2) one can show that for any $\phi \in \Omega^k(\mathbb{R}^d)$, the function ϕ' is alternating and though it is not in general multilinear, it has the property that if $P_i = (1-\lambda) Q_i + \lambda R_i$ for some $\lambda \in \mathbb{R}$ and $Q_i, R_i \in \mathbb{R}^d$ then

$$\phi'(P_0, ..., P_k) = (1-\lambda)\, \phi'(P_0, ..., Q_i, ..., P_k) + \lambda \phi'(P_0, ..., R_i, ..., P_k).$$

Now we derive a lemma which is central to the forthcoming discussion. In analogy to $Q_{k,d}$ let us define $Q^0_{k,n}$ to be the set of $(k+1)$-tuples $\beta = (\beta_0, ..., \beta_k)$ such that $0 \leqslant \beta_0 < \beta_1 < \cdots < \beta_k \leqslant n$.

LEMMA 2.1. *Let $(u_0, ..., u_n)$ be a blending system of functions on $[a, b]$ and let $p(t)$ be the curve in (1.1). Let $\phi \in \Omega^k(\mathbb{R}^d)$ for some k, $1 \leqslant k \leqslant d$. Then for any $s_0, s_1, ..., s_k \in [a, b]$,*

$$\phi'(p(s_0), ..., p(s_k)) = \sum_{\beta \in Q^0_{k,n}} \det M\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} \phi'(P_{\beta_0}, ..., P_{\beta_k}). \quad (2.3)$$

Proof. Let $\alpha \in Q_{k,d}$. Since $\sum_i u_i(t) = 1$ we have the matrix identity

$$\begin{pmatrix} 1 & p^{\alpha_1}(s_0) & \cdots & p^{\alpha_k}(s_0) \\ \vdots & \vdots & & \vdots \\ 1 & p^{\alpha_1}(s_k) & \cdots & p^{\alpha_k}(s_k) \end{pmatrix} = M\begin{pmatrix} u_0, ..., u_n \\ s_0, ..., s_k \end{pmatrix} \begin{pmatrix} 1 & P_0^{\alpha_1} & \cdots & P_0^{\alpha_k} \\ \vdots & \vdots & & \vdots \\ 1 & P_n^{\alpha_1} & \cdots & P_n^{\alpha_k} \end{pmatrix}.$$

By applying the Cauchy–Binet formula (see [1], formula (1.23)) and making the substitution (2.2) we obtain

$$(\phi_\alpha)'\,(p(s_0), ..., p(s_k))$$

$$= \sum_{0 \leqslant \beta_0 < \beta_1 < \cdots < \beta_k \leqslant n} \det M\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} (\phi_\alpha)'\,(P_{\beta_0}, ..., P_{\beta_k}). \quad (2.4)$$

For general $\phi \in \Omega^k(\mathbb{R}^d)$, there exist coefficients $a_\alpha \in \mathbb{R}$ such that $\phi = \sum_{\alpha \in Q_{k,d}} a_\alpha \phi_\alpha$ and therefore

$$\phi' = \sum_{\alpha \in Q_{k,d}} a_\alpha (\phi_\alpha)'.$$

Combining this expression with (2.4) then yields the more general equation (2.3). ∎

There is a parallel expression for rational curves.

LEMMA 2.2. *Let $(u_0, ..., u_n)$ be a blending system on $[a, b]$ and let $p(t)$ be the curve in (1.2). Let $\phi \in \Omega^k(\mathbb{R}^d)$ for some k, $1 \leqslant k \leqslant d$. Then for any $s_0, s_1, ..., s_k \in [a, b]$,*

$$\phi'(p(s_0), ..., p(s_k)) = \frac{1}{w(s_0) \cdots w(s_k)} \sum_{\beta \in Q^0_{k,n}} w_{\beta_0} \cdots w_{\beta_k}$$
$$\times \det M\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} \phi'(P_{\beta_0}, ..., P_{\beta_k}). \quad (2.5)$$

Proof. Let $r_i(t) = w_i u_i(t)/w(t)$. Then $\sum_i r_i(t) = 1$ and so $(r_0, ..., r_n)$ is a blending system. Moreover $p(t) = \sum_i P_i r_i(t)$ and so we can apply

Lemma 2.1 to $p(t)$ with the system $(r_0, ..., r_n)$. Now because determinants are linear functions of rows and columns we have

$$\det M\begin{pmatrix} r_{\beta_0}, ..., r_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} = \frac{w_{\beta_0} \cdots w_{\beta_k}}{w(s_0) \cdots w(s_k)} \det M\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix}$$

and equation (2.5) follows. ∎

3. CONVEXITY PRESERVATION

A planar curve $c: [a, b] \to \mathbb{R}^2$ is said to be *convex* if it crosses any straight line at most twice. It is well known that if $d=2$ and $(u_0, ..., u_n)$ is a totally positive system of blending functions then the curve $p(t)$ in (1.1) is *variation diminishing*, that is the number of times $p(t)$ crosses a straight line l is bounded by the number of times its control polygon $P_0, ..., P_n$ crosses l; see Goodman [9]. An immediate consequence of this property is the classical result that if the control polygon is convex then so is $p(t)$. The variation diminishing property for curves follow from the variation diminishing property of totally positive matrices.

There is an alternative way of defining convexity which is better suited for our purposes. Karlin [10, p. 478], observes that due to Theorem 1.3 of [10], p. 221, if $c(t)$ does not lie on a straight line, it is convex if and only if either

$$\operatorname{vol}(c(s_0), c(s_1), c(s_2)) \geqslant 0, \qquad a \leqslant s_0 < s_1 < s_2 \leqslant b, \tag{3.1}$$

or

$$\operatorname{vol}(c(s_0), c(s_1), c(s_2)) \leqslant 0, \qquad a \leqslant s_0 < s_1 < s_2 \leqslant b. \tag{3.2}$$

Figure 1 shows a curve satisfying (3.1). We say that c is *positively convex* if (3.1) holds and *negatively convex* if (3.2) holds. Moreover we can define a natural generalization of these concepts to arbitrary dimensions.

DEFINITION 3.1. A curve $c: [a, b] \to \mathbb{R}^d$ is *positively d-convex* if $\operatorname{vol}(c(s_0), ..., c(s_d))$ is nonnegative for $a \leqslant s_0 < \cdots < s_d \leqslant b$, and *negatively d-convex* if it is nonpositive. If c is either positively d-convex or negatively d-convex we say that c is d-convex.

Clearly in the case $d=1$, c is a function and is 1-convex if and only if it is monotonic. In the case $d=2$, if the curve c does not line on a straight line, 2-convexity is equivalent to convexity. A 2-convex curve can turn through an angle of up to 2π so 2-convexity is a weaker concept than the

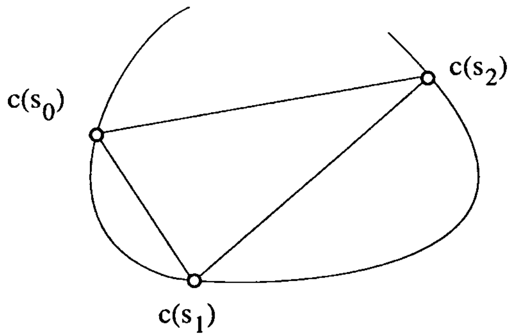

FIG. 1. A convex curve c.

notion of "geometric convexity" introduced in [4] which restricts curves to turn through an angle of at most π.

The following lemma shows how to determine whether a polygonal arc in $\mathbb{R}^d$ is d-convex. To this end we regard the polygonal arc $P_0, ..., P_n$ in $\mathbb{R}^d$ as the parametric piecewise linear curve ψ: $[0, n] \to \mathbb{R}^d$ given by

$$\psi(t) = (i+1-t)\, P_i + (t-i)\, P_{i+1}, \qquad t \in [i, i+1], \quad i = 0, 1, ..., n-1.$$

LEMMA 3.2. *A polygonal arc $P_0, ..., P_n$ in $\mathbb{R}^d$ is positively d-convex if and only if*

$$\operatorname{vol}(P_{\beta_0}, ..., P_{\beta_d}) \geqslant 0, \qquad 0 \leqslant \beta_0 < \cdots < \beta_d \leqslant n. \tag{3.3}$$

Proof. By definition if the polygonal arc $P_0, ..., P_n$ is d-convex then (3.3) is satisfied. Conversely suppose that (3.3) holds. We show that

$$\operatorname{vol}(\psi(s_0), ..., \psi(s_d)) \geqslant 0, \qquad 0 \leqslant s_0 < \cdots < s_d \leqslant n,$$

by induction on the number k of the s_i which are not integers. If $k = 0$ we are done. If $k > 0$ let j be the least index in $\{0, ..., d\}$ for which s_j is not an integer and let $i \in \{0, ..., n-1\}$ be such that $i < s_j < i+1$. Then if $j < d$ and $s_{j+1} < i+1$, we have

$$\psi(s_j) = \frac{s_{j+1} - s_j}{s_{j+1} - i}\, P_i + \frac{s_j - i}{s_{j+1} - i}\, \psi(s_{j+1})$$

and so

$$\operatorname{vol}(\psi(s_0), ..., \psi(s_d))$$
$$= \frac{s_{j+1} - s_j}{s_{j+1} - i} \operatorname{vol}(\psi(s_0), ..., \psi(s_{j-1}), \psi(i), \psi(s_{j+1}), ..., \psi(s_d)) \geqslant 0.$$

Otherwise,

$$\begin{aligned}&\mathrm{vol}(\psi(s_0), ..., \psi(s_d))\\&\quad=(i+1-s_j)\,\mathrm{vol}(\psi(s_0), ..., \psi(s_{j-1}), \psi(i), \psi(s_{j+1}), ..., \psi(s_d))\\&\qquad+(s_j-i)\,\mathrm{vol}(\psi(s_0), ..., \psi(s_{j-1}), \psi(i+1), \psi(s_{j+1}), ..., \psi(s_d))\geqslant 0.\end{aligned}$$ ∎

We now define concepts of d-convexity and rational d-convexity preservation.

Definition 3.3. Let $(u_0, ..., u_n)$ be a blending system of functions on $[a, b]$. If for all positively d-convex control polygons $P_0, ..., P_n$ in $\mathbb{R}^d$, the curve $p(t)$ in (1.1) is positively d-convex, then we say that the system $(u_0, ..., u_n)$ is *d-convexity preserving*. If for all weights $w_0, ..., w_n > 0$ and all positively d-convex control polygons, the curve $p(t)$ in (1.2) is positively d-convex, then we say that the system $(u_0, ..., u_n)$ is *rationally d-convexity preserving*.

We note that (rationally) d-convexity preserving blending systems also preserve negative d-convexity. This follows easily from negating the first coordinate of each control point in (1.1) or (1.2).

Our immediate goal is to characterize rationally d-convexity preserving blending systems. We establish the essential part of the characterization in the following.

Proposition 3.4. *Let $(u_0, ..., u_n)$ be a blending system on $[a, b]$. For all d, $1 \leqslant d \leqslant n$, the system $(u_0, ..., u_n)$ is rationally d-convexity preserving if and only if it is* WT_{d+1}.

Proof. Letting $k = d$ and $\phi = \det / d!$ in (2.5) we have that if $p(t)$ is the curve in (1.2),

$$\begin{aligned}&\mathrm{vol}(p(s_0), ..., p(s_d))\\&\quad=\frac{1}{w(s_0)\cdots w(s_d)}\sum_{\beta\in Q^0_{d,n}} w_{\beta_0}\cdots w_{\beta_d}\\&\qquad\times\det M\begin{pmatrix}u_{\beta_0}, ..., u_{\beta}\\ s_0, ..., s_d\end{pmatrix}\mathrm{vol}(P_{\beta_0}, ..., P_{\beta_d})\end{aligned} \tag{3.4}$$

for any $s_0, s_1, ..., s_d \in [a, b]$.

Suppose that $(u_0, ..., u_n)$ is WT_{d+1}. Let $w_0, ..., w_n > 0$ and let $P_0, ..., P_n$ be a positively d-convex control polygon in $\mathbb{R}^d$. Then from (3.4), we have that $\mathrm{vol}(p(s_0), ..., p(s_d)) \geqslant 0$ provided $s_0 < s_1 < \cdots < s_d$ in $[a, b]$ and so $(u_0, ..., u_n)$ is rationally d-convexity preserving.

For the converse suppose that $(u_0, ..., u_n)$ is rationally d-convexity preserving. Let $\gamma \in Q^0_{d,n}$ and let $s_0 < \cdots < s_d$ be an increasing sequence in $[a, b]$. Choose $P_0 = \cdots = P_{\gamma_0} = 0$, and for $j = 0, ..., d-2$, choose $P_{\gamma_j+1} = \cdots = P_{\gamma_{j+1}} = e_{j+1}$ and let $P_{\gamma_{d-1}+1} = \cdots = P_n = e_d$. Then the points $P_0, ..., P_n$ are all vertices of the standard d-simplex in $\mathbb{R}^d$ and using (2.1), we have that for any $\beta \in Q^0_{d,n}$,

$$\operatorname{vol}(P_{\beta_0}, ..., P_{\beta_d}) = \begin{cases} 1/d!, & P_{\beta_0}, ..., P_{\beta_d} \text{ are pairwise distinct;} \\ 0, & \text{otherwise.} \end{cases}$$

In particular $P_{\gamma_0} = 0$ and $P_{\gamma_j} = e_j$ for $j = 1, ..., d$, and so $\operatorname{vol}(P_{\gamma_0}, ..., P_{\gamma_d}) = 1/d!$. Since the control polygon $P_0, ..., P_n$ is positively d-convex, we have from (3.4) that

$$\sum_{\beta \in Q^0_{d,n}} w_{\beta_0} \cdots w_{\beta_d} \det M \begin{pmatrix} u_{\beta_0}, ..., u_{\beta_d} \\ s_0, ..., s_d \end{pmatrix} \operatorname{vol}(P_{\beta_0}, ..., P_{\beta_d}) \geqslant 0.$$

The final step is to employ a technique used in [6] and [2]: we let $w_i = 1$ for $i \in \{\gamma_0, ..., \gamma_d\}$ and $w_i = \varepsilon$ otherwise. In the limit as $\varepsilon \to 0$ we deduce that

$$0 \leqslant \det M \begin{pmatrix} u_{\gamma_0}, ..., u_{\gamma_d} \\ s_0, ..., s_d \end{pmatrix} \operatorname{vol}(P_{\gamma_0}, ..., P_{\gamma_d}) = \frac{1}{d!} \det M \begin{pmatrix} u_{\gamma_0}, ..., u_{\gamma_d} \\ s_0, ..., s_d \end{pmatrix}$$

and therefore $(u_0, ..., u_n)$ is WT_{d+1}. ∎

By applying Proposition 3.4 for all d, $1 \leqslant d \leqslant n$, we immediately deduce:

COROLLARY 3.5. *A blending system $(u_0, ..., u_n)$ on $[a, b]$ is totally positive if and only if it is rationally d-convexity preserving for all $d = 1, ..., n$.*

More generally, $(u_0, ..., u_n)$ is TP_{k+1}, $1 \leqslant k \leqslant n$, if and only if it is rationally d-convexity preserving for all $d = 1, ..., k$.

Next we consider d-convexity preserving systems. Since a rationally d-convexity preserving blending system is also d-convexity preserving, we have from Proposition 3.4 that a sufficient condition for a blending system to be d-convexity preserving is that it is WT_{d+1}.

For the remainder of this section we concentrate on the case when $p(t)$ in (1.1) is a planar curve. We say that a system of functions is *convexity preserving* if it is 2-convexity preserving and so a sufficient condition for convexity preservation is WT_3. Now we derive a necessary condition. Given functions $u_0, ..., u_n$ on $[a, b]$, let us define the functions

$$v_i = \sum_{j=i}^{n} u_j, \qquad i = 0, 1, 2, ..., n,$$

on $[a, b]$ and note that if $(u_0, ..., u_n)$ is a blending system then $v_0(t) = 1$.

PROPOSITION 3.6. *Let* $(u_0, ..., u_n)$ *be a system of blending functions on* $[a, b]$. *If* $(u_0, ..., u_n)$ *is convexity preserving then the system* $(1, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2})$ *is weak Tchebycheff whenever* $0 \leqslant j_0 < j_1 < j_2 \leqslant n$.

Proof. Letting $k = 2$ and $\phi = \det/2$ in (2.3) we have for $p(t)$ in (1.1) and $s_0, s_1, s_2 \in [a, b]$,

$$\operatorname{vol}(p(s_0), p(s_1), p(s_2)) = \sum_{0 \leqslant i_0 < i_1 < i_2 \leqslant n} \det M\begin{pmatrix} u_{i_0}, u_{i_0}, u_{i_2} \\ s_0, s_1, s_2 \end{pmatrix} \operatorname{vol}(P_{i_0}, P_{i_1}, P_{i_2}). \tag{3.5}$$

Suppose that $(u_0, ..., u_n)$ is convexity preserving and let $0 \leqslant j_0 < j_1 < j_2 \leqslant n$ and $s_0 < s_1 < s_2$ in $[a, b]$. Choose $P_0 = \cdots = P_{j_0 - 1} = (0, 0)$ (if $j_0 > 0$), $P_{j_0} = \cdots = P_{j_1 - 1} = (1, 0)$, $P_{j_1} = \cdots = P_{j_2 - 1} = (0, 1)$, and $P_{j_2} = \cdots = P_n = (0, 0)$. Then the control polygon $P_0, ..., P_n$ is positively convex and so $\operatorname{vol}(p(s_0), p(s_1), p(s_2)) \geqslant 0$. Moreover, if i_0, i_1, i_2 satisfy $0 \leqslant i_0 < i_1 < i_2 \leqslant n$, we find that $\operatorname{vol}(P_{i_0}, P_{i_1}, P_{i_2}) = 1/2$ when either $0 \leqslant i_0 < j_0 \leqslant i_1 < j_1 \leqslant i_2 < j_2$ or $j_0 \leqslant i_0 < j_1 \leqslant i_1 < j_2 \leqslant i_2 \leqslant n$ and $\operatorname{vol}(P_{i_0}, P_{i_1}, P_{i_2}) = 0$ otherwise. Therefore from (3.5) we deduce the inequality

$$\sum_{\substack{0 \leqslant i_0 < j_0 \\ j_0 \leqslant i_1 < j_1 \\ j_1 \leqslant i_2 < j_2}} \det M\begin{pmatrix} u_{i_0}, u_{i_1}, u_{i_2} \\ s_0, s_1, s_2 \end{pmatrix} + \sum_{\substack{j_0 \leqslant i_0 < j_1 \\ j_1 \leqslant i_1 < j_2 \\ j_2 \leqslant i_2 \leqslant n}} \det M\begin{pmatrix} u_{i_0}, u_{i_1}, u_{i_2} \\ s_0, s_1, s_2 \end{pmatrix} \geqslant 0. \tag{3.6}$$

Using standard properties of determinants, we can then rewrite the left hand side of (3.6) as

$$\begin{aligned} &\det M\begin{pmatrix} 1 - v_{j_0}, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2} \\ s_0, s_1, s_2 \end{pmatrix} + \det M\begin{pmatrix} v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2}, v_{j_2} \\ s_0, s_1, s_2 \end{pmatrix} \\ &\quad = \det M\begin{pmatrix} 1 - v_{j_2}, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2} \\ s_0, s_1, s_2 \end{pmatrix} + \det M\begin{pmatrix} v_{j_2}, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2} \\ s_0, s_1, s_2 \end{pmatrix} \\ &\quad = \det M\begin{pmatrix} 1, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2} \\ s_0, s_1, s_2 \end{pmatrix}. \quad \blacksquare \end{aligned}$$

It was shown in Theorem 3.5 of [4] that under certain assumptions, a system $(u_0, ..., u_n)$ is "geometrically convexity preserving" if and only if it satisfies the condition that the systems of three functions $(1, v_i, v_j)$ are weak Tchebycheff for all i, j, $1 \leqslant i < j \leqslant n$. As one might expect, the necessary condition of Proposition 3.6 is stronger. To see this we observe that when $j_0 = 0$ the system $(1, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2})$ is weak Tchebycheff if and only if the system $(1, v_{j_1}, v_{j_2})$ is weak Tchebycheff.

Let us consider the systems $(1, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2})$ for $0 \leqslant j_0 < j_1 < j_2 \leqslant n$ in the two cases $n = 2, 3$. In the former case these is only one such system which is determined by $(j_0, j_1, j_2) = (0, 1, 2)$ and it is weak Tchebycheff if and only if the system (u_0, u_1, u_2) is weak Tchebycheff. The latter case, $n = 3$, reveals the cylic nature of the condition in Proposition 3.6. There are four possible choices of (j_0, j_1, j_2), namely $(1, 2, 3)$, $(0, 2, 3)$, $(0, 1, 3)$, and $(0, 1, 2)$ and the four systems $(1, v_{j_0} - v_{j_1}, v_{j_1} - v_{j_2})$ are weak Tchebycheff if and only if the four systems

$$(u_3 + u_0, u_1, u_2), \quad (u_3, u_0 + u_1, u_2), \quad (u_3, u_0, u_1 + u_2), \quad (u_2 + u_3, u_0, u_1) \tag{3.7}$$

are weak Tchebycheff respectively.

By showing that the necessary condition of Proposition 3.6 is sufficient when $n = 2$ or 3 we thus obtain the following characterization of convexity preservation.

COROLLARY 3.7. *A system (u_0, u_1, u_2) on $[a, b]$ is convexity preserving if and only if it is weak Tchebycheff. A system (u_0, u_1, u_2, u_3) on $[a, b]$ is convexity preserving if and only if the four systems (3.7) are weak Tchebycheff.*

Proof. When $n = 2$ it is immediate from Proposition 3.4 that if (u_0, u_1, u_2) is weak Tchebycheff then it is also convexity preserving. In the case $n = 3$ suppose that the systems (3.7) are weak Tchebycheff and that P_0, P_1, P_2, P_3 is a positively convex control polygon (see Fig. 2). For convenience let us identify u_{j+4k} (resp. P_{j+4k}) with u_j (resp. P_j) for $j = 0, 1, 2, 3$ and $k \in \mathbb{Z}$ and we let

$$D_i(s) = \det M \begin{pmatrix} u_i, u_{i+1}, u_{i+2} \\ s_0, s_1, s_2 \end{pmatrix}, \qquad s = (s_0, s_1, s_2), \quad i \in \mathbb{Z}.$$

By geometrical considerations we see that for any $i \in \mathbb{Z}$, the intersection of the two (possibly degenerate) triangles $\triangle P_{i-1}, P_i, P_{i+1}$ and $\triangle P_i, P_{i+1}, P_{i+2}$ is a third (possibly degenerate) triangle whose area we denote by A_i; see Fig. 2. Then from (3.5) we find for $s_0 < s_1 < s_2$ in $[a, b]$, that

$$\begin{aligned} \operatorname{vol}(p(s_0), p(s_1), p(s_2)) &= \sum_{i=0}^{3} D_i(s) \operatorname{vol}(P_i, P_{i+1}, P_{i+2}) \\ &= \sum_{i=0}^{3} D_i(s)(A_i + A_{i+1}) \\ &= \sum_{i=0}^{3} (D_i(s) + D_{i-1}(s))\, A_i \geqslant 0, \end{aligned}$$

and so the curve $p(t)$ is positively convex. ∎

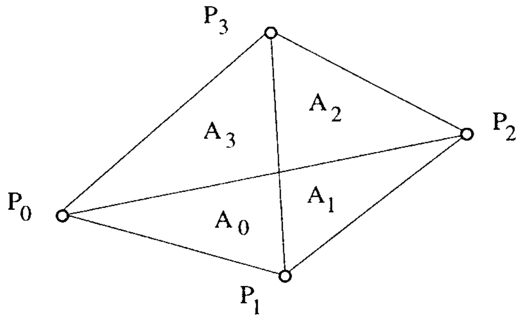

FIG. 2. Areas of triangles formed by the polygon.

4. CURVE INTERSECTIONS

In this section we study normal vectors and intersections between curves. Let us define the *convex cone* of a set of vectors S to be

$$\langle S\rangle_+ := \left\{ \sum_{i=1}^{n} \lambda_i c_i \mid \lambda_i \geqslant 0,\, c_i \in S,\, n \geqslant 1 \right\}.$$

It was shown in [2] that if $(u_0, ..., u_n)$ is totally positive system of blending functions then a curve $p(t)$ in $\mathbb{R}^3$ of the form (1.1) has the property that for $a \leqslant s_0 < s_1 < s_2 \leqslant b$, the normal (or binormal) vector

$$n'(p(s_0), p(s_1), p(s_2)) = (p(s_1) - p(s_0)) \times (p(s_2) - p(s_1))$$

belongs to a cone of vectors generated by the control polygon, namely

$$n'(p(s_0), p(s_1), p(s_2)) \in \langle (P_i - P_{i-1}) \times (P_j - P_{j-1}) \mid 0 < i < j \leqslant n\rangle_+. \tag{4.1}$$

Using this fact, it was further shown in Proposition 5.5 of [2] that if the convex cones of two curves generated by totally positive blending systems intersect only at the origin then the curves intersect in at most two non-collinear points. Letting $\phi = n^i$, for $i = 1, 2, 3$ and $k = 2$, $d = 3$ in Lemma 2.1, however, we find that the normal vector belongs to a smaller convex cone, indeed

$$n'(p(s_0), p(s_1), p(s_2)) \in \langle (P_j - P_i) \times (P_k - P_j) \mid 0 \leqslant i < j < k \leqslant n\rangle_+. \tag{4.2}$$

The cone in (4.2) is a subset of the cone in (4.1) because

$$(P_j - P_i) \times (P_k - P_j) = \sum_{r=i+1}^{j} \sum_{s=j+1}^{k} (P_r - P_{r-1}) \times (P_s - P_{s-1}).$$

Moreover again using Lemma 2.1, (4.2) generalizes to arbitrary dimensions:

PROPOSITION 4.1. *Let $p(t)$ be the curve in (1.1) where $(u_0, ..., u_n)$ is a $\mathrm{WT_d}$ blending system on $[a, b]$. If $s_0 < s_1 < \cdots < s_{d-1}$ in $[a, b]$ then*

$$n'(p(s_0), ..., p(s_{d-1})) \in \langle n'(P_{\beta_0}, ..., P_{\beta_{d-1}}) \mid \beta \in Q^0_{d-1,n} \rangle_+.$$

We can apply Proposition 4.1 in order to bound the number of intersections between two curves of the form (1.1).

COROLLARY 4.2. *Let $P_0, ..., P_n,\ Q_0, ..., Q_m \in \mathbb{R}^d$. Let $p(t) = \sum P_i f_i(t)$, and $q(s) = \sum Q_i g_i(s)$ be the curves generated by two $\mathrm{WT_d}$ blending systems $(f_0, ..., f_n)$ and $(g_0, ..., g_m)$ on $[a_1, b_1]$ and $[a_2, b_2]$ respectively. Let C and D be the two convex cones*

$$C = \langle n'(P_{\beta_0}, ..., P_{\beta_{d-1}}) \mid \beta \in Q^0_{d-1,n} \rangle_+,$$

$$D = \langle n'(Q_{\beta_0}, ..., Q_{\beta_{d-1}}) \mid \beta \in Q^0_{d-1,m} \rangle_+,$$

and suppose that $C \cap D = C \cap (-D) = \{0\}$. Then if the curves $p(t)$ and $q(s)$ intersect in d points then those d points are contained in a hyperplane of dimension $d-2$.

Proof. Suppose in order to get a contradiction that $p(t_i) = q(s_i)$ for $i = 0, 1, ..., d-1$ with $t_0 < t_1 < \cdots < t_{d-1}$ and that $p(t_0), ..., p(t_{d-1})$ are the vertices of a (non-degenerate) $(d-1)$-simplex. Then the normal vectors

$$n_1 = n'(p(t_0), ..., p(t_{d-1})), \qquad \text{and} \qquad n_2 = n'(q(s_0), ..., q(s_{d-1}))$$

are non-zero and either $n_2 = n_1$ or $n_2 = -n_1$. Moreover from Proposition 4.1, $n_1 \in C$ and either $n_2 \in D$ or $n_2 \in -D$ depending on whether the sequence $s_0, ..., s_{d-1}$ is an even or odd permutation respectively of its ordering in increasing sequence. Therefore either $C \cap D \neq \{0\}$ or $C \cap (-D) \neq \{0\}$, which is a contradiction. ∎

5. DERIVATIVES AND WRONSKIANS

In this section we express alternating functions of derivatives of curves in terms of Wronskians using Lemma 2.2. We only treat the more general rational curves (1.2) since the non-rational curve (1.1) is the special case of (1.2) with equal weights w_i.

For distinct $t_0, ..., t_i$ in $\mathbb{R}$, let $u[t_0, ..., t_i]$ denote the usual ith divided difference of a function u defined by

$$u[t_0] = u(t_0), \tag{5.1}$$

$$u[t_0, ..., t_i] = \frac{u[t_1, ..., t_i] - u[t_0, ..., t_{i-1}]}{t_i - t_0} \tag{5.2}$$

and for a sequence of functions $u_0, ..., u_n$, let

$$H\begin{pmatrix} u_0, ..., u_n \\ t_0, ..., t_m \end{pmatrix} = (u_j[t_0, t_1, ..., t_i])_{0 \leqslant i \leqslant m,\, 0 \leqslant j \leqslant n}.$$

If $u_0, ..., u_n$ are C^n we also define their Wronskian matrix

$$W(u_0, ..., u_n)(t) = (u_j^{(i)}(t))_{0 \leqslant i \leqslant n,\, 0 \leqslant j \leqslant n}.$$

It will be useful in subsequent discussions to define for $k = 0, 1, ..., m$ the constant

$$R_{m,k} = m^k (m-1)^{k-1} \cdots (m-k+1)$$

and we note that

$$R_{k,k} = k!\,(k-1)! \cdots 2!\,1!.$$

PROPOSITION 5.1. *Suppose that* $(u_0, ..., u_n)$ *is a blending system of functions in* $C^k[a, b]$. *Let* $\phi \in \Omega^k(\mathbb{R}^d)$ *and* $p(t)$ *be as in* (1.2). *For any* $t \in [a, b]$,

$$\phi(p'(t), ..., p^{(k)}(t)) = \frac{1}{(w(t))^{k+1}} \sum_{\beta \in Q^0_{k,n}} w_{\beta_0} \cdots w_{\beta_k} \det W(u_{\beta_0}, ..., u_{\beta_k})(t)\, \phi'(P_{\beta_0}, ..., P_{\beta_k}). \tag{5.3}$$

Proof. Applying (5.2) inductively one can show that for any $s_0, ..., s_k$ in $[a, b]$,

$$\phi'(p(s_0), ..., p(s_k)) = \prod_{0 \leqslant i < j \leqslant k} (s_j - s_i)\, \phi(p[s_0, s_1], ..., p[s_0, s_1, ..., s_k])$$

and a similar argument using (5.2) shows that

$$\det M\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} = \prod_{0 \leqslant i < j \leqslant k} (s_j - s_i) \det H\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix}.$$

Substituting these expressions into (2.5) we obtain

$$\begin{aligned} &\phi(p[s_0, s_1], ..., p[s_0, s_1, ..., s_k]) \\ &\quad = \frac{1}{w(s_0) \cdots w(s_k)} \sum_{\beta \in Q^0_{k,n}} w_{\beta_0} \cdots w_{\beta_k} \\ &\quad\quad \times \det H\begin{pmatrix} u_{\beta_0}, ..., u_{\beta_k} \\ s_0, ..., s_k \end{pmatrix} \phi'(P_{\beta_0}, ..., P_{\beta_k}). \end{aligned} \tag{5.4}$$

Multiplying each side of (5.4) by a factor of $R_{k,k}$, letting $s_0, ..., s_k \to t$, and recalling that $u[t_0, ..., t_i]$ converges to $u^{(i)}(t)/i!$, we obtain in the limit (5.3). ▮

Letting $k = d$ and $\phi = \det$ in Proposition 5.1, it follows that if the blending system $(u_0, ..., u_n)$ is totally positive then the highest order curvature κ_{d-1} of the curve $p(t)$ in (1.2) is nonnegative provided that the control polygon $P_0, ..., P_n$ is positively d-convex. For letting $s_0, ..., s_k$ converge to t with the constraint that $s_0 < \cdots < s_k$ we see that every Wronskian in (5.3) has nonnegative determinant.

6. BERNSTEIN POLYNOMIALS

A common example of a blending system on $[0, 1]$ is the Bernstein basis $(B_{0,n}, ..., B_{n,n})$ of the space π_n of polynomials of degree $\leqslant n$,

$$B_{i,n}(t) := \binom{n}{i} t^i (1-t)^{n-i}, \qquad t \in [0, 1].$$

In this case, the curve $p(t)$ in (1.1) is called a *Bézier curve* and the curve in (1.2) *is a rational Bézier curve.*

In this section we study the Bernstein basis and we make the convention that $\binom{n}{i} = 0$ when $i < 0$ or $i < n$. Let $A_{s,n}$ be the rectangular $(s+1) \times (n+s+1)$ matrix

$$A_{s,n}(t) = (B_{j-i,n}(t))_{0 \leqslant i \leqslant s,\, 0 \leqslant j \leqslant n+s}$$

which is banded with band width $n+1$. For $\beta \in Q^0_{s,n}$, the columns of $A_{s,n}$ corresponding to $\beta_0+1, ..., \beta_s+1$ form a square $(s+1)\times(s+1)$ submatrix

$$A_n(\beta_0, ..., \beta_s) := A_{s,n}\begin{pmatrix} 1, ..., s+1 \\ \beta_0+1, ..., \beta_s+1 \end{pmatrix} = \begin{pmatrix} B_{\beta_0,n} & \cdots & B_{\beta_s,n} \\ \vdots & & \vdots \\ B_{\beta_0-s,n} & \cdots & B_{\beta_s-s,n} \end{pmatrix}.$$

LEMMA 6.1. *For any k, $0 \leqslant k \leqslant n$, let $\beta \in Q^0_{k,n}$. Then for all $t \in [a, b]$,*

$$\det W(B_{\beta_0,n}, B_{\beta_1,n}, ..., B_{\beta_k,n})(t) = R_{n,k} \det A_{n-k}(\beta_0, ..., \beta_k)(t). \quad (6.1)$$

Proof. The proof is by induction on k. Since $R_{n,0}=1$, Eq. (6.1) holds when $k=0$. Let $k>0$ and suppose that (6.1) holds when k is replaced by $k-1$. Then because a determinant is a linear combination of the elements of its last row, we have

$$\det W(B_{\beta_0,n}, ..., B_{\beta_k,n}) = R_{n,k-1}\begin{vmatrix} B_{\beta_0,n-k+1} & \cdots & B_{\beta_k,n-k+1} \\ \vdots & & \vdots \\ B_{\beta_0-k+1,n-k+1} & \cdots & B_{\beta_k-k+1,n-k+1} \\ B^{(k)}_{\beta_0,n} & \cdots & B^{(k)}_{\beta_k,n} \end{vmatrix}. \quad (6.2)$$

Now we express every element of the last row of the determinant on the right hand side of (6.2) as a linear combination of Bernstein polynomials of lower degree using the identity

$$B^{(k)}_{i,n}(t) = \frac{n!}{(n-k)!}\,\delta^k B_{i,n-k}(t),$$

where δ is the backward difference operator, $\delta B_{j,m} = B_{j-1,m} - B_{j,m}$. The right hand side of (6.2) then becomes $R_{n,k} \det A$ where

$$A = \begin{pmatrix} B_{\beta_0,n-k+1} & \cdots & B_{\beta_k,n-k+1} \\ \vdots & & \vdots \\ B_{\beta_0-k+1,n-k+1} & \cdots & B_{\beta_k-k+1,n-k+1} \\ \delta^k B_{\beta_0,n-k} & \cdots & \delta^k B_{\beta_k,n-k} \end{pmatrix}.$$

Let $a_1, ..., a_{k+1}$ be the row vectors of A. The determinant of A is unchanged if we replace row a_k by

$$\hat{a}_k := a_k + \sum_{j=0}^{k-2} (-1)^{k-1-j}\binom{k-1}{j} a_{j+1} - t a_{k+1}.$$

Noting the identity

$$\delta^{k-1} B_{j,m+1}(t) = \delta^{k-1}((1-t)\, B_{j,m}(t) + t B_{j-1,m}(t))$$
$$= \delta^{k-1} B_{j,m}(t) + t\delta^k B_{j,m}(t),$$

it follows that the $(i+1)$th element of $\hat{a}_k$ is

$$\delta^{k-1} B_{\beta_i, n-k+1} - t\delta^k B_{\beta_i, n-k} = \delta^{k-1} B_{\beta_i, n-k}.$$

After the substitution of a_k by $\hat{a}_k$ we further substitute row a_{k-1} by

$$a_{k-1} + \sum_{j=0}^{k-3} (-1)^{k-2-j} \binom{k-2}{j} a_{j+1} - t a_k.$$

Continuing in this way until row a_1 has been substituted we see that

$$\det A = \begin{vmatrix} B_{\beta_0, n-k} & \cdots & B_{\beta_k, n-k} \\ B_{\beta_0 - 1, n-k} - B_{\beta_0, n-k} & \cdots & B_{\beta_k - 1, n-k} - B_{\beta_k, n-k} \\ \vdots & & \vdots \\ \delta^{k-1} B_{\beta_0, n-k} & \cdots & \delta^{k-1} B_{\beta_k, n-k} \\ \delta^k B_{\beta_0, n-k} & \cdots & \delta^k B_{\beta_k, n-k} \end{vmatrix}.$$

We now add row a_1 to row a_2 so that a_2 becomes

$$(B_{\beta_0 - 1, n-k}, ..., B_{\beta_k - 1, n-k}).$$

We then add $-a_1 + 2a_2$ to row a_3 and so on until we have replaced row a_{k+1} by which time we have established that

$$\det A = \det A_{n-k}(\beta_0, ..., \beta_k). \quad \blacksquare$$

Substituting the expression for the Wronskian in (6.1) into equation (5.3) we can express $\phi(p'(t), ..., p^{(k)}(t))$ in terms of Bernstein polynomials of degree $n-k$:

$$\phi(p'(t), ..., p^{(k)}(t))$$
$$= \frac{R_{n,k}}{(w(t))^{k+1}} \sum_{\beta \in Q^0_{k,n}} w_{\beta_0} \cdots w_{\beta_k} \det A_{n-k}(\beta_0, ..., \beta_k)(t)\, \phi'(P_{\beta_0}, ..., P_{\beta_k}). \tag{6.3}$$

Now we wish to consider de Casteljau's algorithm for the evaluation of Bézier curves. We will only treat the more general rational de Casteljau algorithm [7] in which one defines the functions $w_{i,r}(t)$ and $p_{i,r}(t)$ for $r=0, ..., n$, $i=0, ..., r$ by

$$w_{i,0}(t)=w_i, \qquad p_{i,0}(t)=P_i,$$

and for $r=1, ..., n$ by the two triangular schemes

$$w_{i,r}(t)=(1-t)\,w_{i,r-1}(t)+tw_{i+1,r-1}(t),$$
$$p_{i,r}(t)=((1-t)\,w_{i,r-1}(t)\,p_{i,r-1}(t)+tw_{i+1,r-1}(t)\,p_{i+1,r-1}(t))/w_{i,r}(t).$$

It can be shown that both $w(t)=w_{0,n}(t)$ and $p(t)=p_{0,n}(t)$ and

$$w_{i,s}(t)=\sum_{j=0}^{s-r} w_{i+j,r}(t)\,B_{j,s-r}(t), \tag{6.4}$$

$$p_{i,s}(t)=\sum_{j=0}^{s-r} w_{i+j,r}(t)\,p_{i+j,r}(t)\,B_{j,s-r}(t)/w_{i,s}(t), \tag{6.5}$$

for $0\leqslant r\leqslant s\leqslant n$ and $i=0, ..., s$.

Lemma 6.2. *Let $p(t)$ in (1.2) be a rational Bézier curve. Let $\phi\in\Omega^k(\mathbb{R}^d)$ for some k, $1\leqslant k\leqslant d$. Then for any $r=0, ..., n-k$ and $t\in[0,1]$,*

$$\begin{aligned} w_{0,n-k}(t)\cdots w_{k,n-k}(t)\,\phi'(p_{0,n-k}(t), ..., p_{k,n-k}(t)) \\ = \sum_{\beta\in Q^0_{k,n-r}} w_{\beta_0,r}(t)\cdots w_{\beta_k,r}(t)\det A_{n-k-r}(\beta_0, ..., \beta_k)(t) \\ \times\phi'(p_{\beta_0,r}(t), ..., p_{\beta_k,r}(t)). \end{aligned} \tag{6.6}$$

Proof. Let $\alpha\in Q_{k,d}$. For $s=0, 1, ..., n-k$, let $P_{s,n,\alpha}$ be the $(n-s+1)\times(k+1)$ matrix

$$P_{s,n,\alpha}(t)=\begin{pmatrix} w_{0,s}(t) & w_{0,s}(t)\,p^{\alpha_1}_{0,s}(t)\cdots & w_{0,s}(t)\,p^{\alpha_k}_{0,s}(t) \\ \vdots & \vdots & \vdots \\ w_{n-s,s}(t) & w_{n-s,s}(t)\,p^{\alpha_1}_{n-s,s}(t)\cdots & w_{n-s,s}(t)\,p^{\alpha_k}_{n-s,s}(t) \end{pmatrix}$$

where $p^1_{i,s}(t), ..., p^d_{i,s}(t)$ are the coordinates of the point $p_{i,s}(t)\in\mathbb{R}^d$. Then from (6.4) and (6.5) we have the matrix identity

$$P_{n-k,n,\alpha}(t)=A_{k,n-k-r}(t)\,P_{r,n,\alpha}(t).$$

We apply the Cauchy–Binet formula to this equation and using the fact that determinants are linear functions of columns and recalling (2.2) we obtain equation (6.6) in the case $\phi=\phi_\alpha$. Since the ϕ_α, $\alpha\in Q_{k,d}$, form a basis for $\Omega^k(\mathbb{R}^d)$, Eq. (6.6) therefore holds for any $\phi\in\Omega^k(\mathbb{R}^d)$. ∎

Letting $r=0$ in (6.6) we obtain

$$w_{0,n-k}(t)\cdots w_{k,n-k}(t)\,\phi'(p_{0,n-k}(t),...,p_{k,n-k}(t))$$
$$=\sum_{\beta\in Q^0_{k,n}} w_{\beta_0}\cdots w_{\beta_k}A_{n-k}(\beta_0,...,\beta_k)(t)\,\phi'(P_{\beta_0},...,P_{\beta_k})$$

and substituting this into (6.3) we finally obtain an expression for $\phi(p'(t),...,p^{(k)}(t))$ in terms of the points and weights of the de Casteljau algorithm of level $n-k$:

THEOREM 6.3. *Under the hypothesis of Lemma* 6.2,

$$\phi(p'(t),...,p^{(k)}(t))$$
$$=R_{n,k}\,\frac{w_{0,n-k}(t)\cdots w_{k,n-k}(t)}{(w(t))^{k+1}}\,\phi'(p_{0,n-k}(t),...,p_{k,n-k}(t)). \quad (6.7)$$

Equation (6.7) generalizes some formulas presented in [8] which were derived there by direct differentiation. For example, by letting $k=1$ and considering the d functions $\phi_1,...,\phi_d\in\Omega^1(\mathbb{R}^d)$, $\phi_i(v)=v^i$, where $v=(v^1,...,v^d)^T$, equation (6.7) leads to

$$p'=n\,\frac{w_{0,n-1}w_{1,n-1}}{w^2}\,(p_{1,n-1}-p_{0,n-1}). \quad (6.8)$$

In the case $d=3$, since each coordinate function of the cross product belongs to $\Omega^2(\mathbb{R}^3)$ we find

$$p'\times p''=n^2(n-1)\,\frac{w_{0,n-2}w_{1,n-2}w_{2,n-2}}{w^3}\,(p_{1,n-k}-p_{0,n-k})$$
$$\times(p_{2,n-k}-p_{1,n-k}), \quad (6.9)$$

and letting $\phi=\det\in\Omega^3(\mathbb{R}^3)$ we obtain

$$(p'\times p'')\cdot p'''=n^3(n-1)^2\,(n-2)\,\frac{w_{0,n-3}\cdots w_{3,n-3}}{w^4}$$
$$\times((p_{1,n-3}-p_{0,n-3})\times(p_{2,n-3}-p_{1,n-3}))$$
$$\cdot(p_{3,n-3}-p_{2,n-3}). \quad (6.10)$$

Equations (6.8–6.10) provide a numerically stable way of computing the curvature and torsion of the curve $p(t)$ using the expressions

$$\kappa(t) = \frac{\|p'(t) \times p''(t)\|}{\|p'(t)\|^3}, \qquad \tau(t) = \frac{(p'(t) \times p''(t)) \cdot p'''(t)}{\|p'(t) \times p''(t)\|^2}.$$

Combining (6.7) with (6.6) for general r we can express $\phi(p'(t), ..., p^{(k)}(t))$ more generally in terms of points of the de Casteljau algorithm of any level $r = 0, ..., n-k$:

$$\phi(p'(t), ..., p^{(k)}(t)) = \frac{R_{n,k}}{(w(t))^{k+1}} \sum_{\beta \in Q^0_{k,n-r}} w_{\beta_0,r}(t) \cdots w_{\beta_k,r}(t) \times \det A_{n-k-r}(\beta_0, ..., \beta_k)(t)\, \phi'(p_{\beta_0,r}(t), ..., p_{\beta_k,r}(t)).$$

This equation reduces to (6.3) and (6.7) when $r = 0$ and $r = n-k$ respectively.

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